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Berry curvature and 4-dimensional monopole in relativistic chiral kinetic equation

Jiunn-Wei Chen,[1] Shi Pu,[1,2] Qun Wang,[2] and Xin-Nian Wang[3,4]

[1] *Department of Physics, Center for Theoretical Sciences, and Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei 10617, Taiwan*

[2] *Interdisciplinary Center for Theoretical Study and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China*

[3] *Key Laboratory of Quark and Lepton Physics (MOE) and Institute of Particle Physics, Central China Normal University, Wuhan, 430079, China*

[4] *Nuclear Science Division, MS 70R0319, Lawrence Berkeley National Laboratory, Berkeley, California 94720*

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We derive a relativistic chiral kinetic equation with manifest Lorentz covariance and a 4-dimensional Euclidean Berry monopole. The theory is based on Wigner functions of spin-1/2 massless fermions in a constant electromagnetic background. By integrating out the p_0-component of the 4-momentum p, the previous 3-dimensional results derived (without vorticity) from the Hamiltonian approach is reproduced. The phase space continuity equation has a source term proportional to the product of the vorticity and electric field while the axial anomaly arises from the flux of the monopole. This makes the chiral magnetic effect, vorticity, chiral anomaly, Berry phase, and the monopole can all be described in a unified way by Wigner functions.

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Introduction. — Chiral anomaly is an important quantum effect which is absent at the classical level. Recently it has been shown that such a microscopic quantum effect can have a macroscopic impact on the dynamics of relativistic fluids. The chiral magnetic and vortical effect (CME and CVE) [1–3] are manifested as currents induced by magnetic field and vorticity. Such effects and related topics have been investigated within a variety of approaches, such as AdS/CFT correspondence [4–8], relativistic hydrodynamics [9–11], and quantum field theory [2, 12–17]. Derivation of a semi-classical chiral kinetic equation incorporating CME and CVE will deepen our understanding of the underlying mechanism of these effects. Recently it has been found that features of Berry phase due to a 3-dimensional momentum monopole emerge in a chiral kinetic equation without manifest Lorentz covariance [18, 19]. A semi-classical kinetic equation has also been derived in an electron system with Berry curvature [20]. The Berry phase is a topological phase acquired by an energy eigenstate after it undergoes an adiabatic evolution along a loop in parameter space [21]. The Aharonov-Bohm effect is an example of the Berry phase in electromagnetism. The Berry phase has been applied to a variety of condensed matter phenomena, such as the quantum Hall effect, the spin Hall effect, etc. For a recent review, see e.g. Ref. [22].

In this paper we will derive a new chiral kinetic equation with manifest Lorentz covariance from the Wigner function [24]. We will show that such a chiral kinetic equation incorporates features of the Berry curvature and 4-dimensional Euclidean monopole. These results reveal the inherent connection between the Berry phase and gauge invariant Wigner functions. One advantage of our approach is that the vorticity effect can be derived straightforwardly which, apparently, is not the case in other approaches. The non-covariant version of our covariant equation is shown to be in agreement with previous results if vorticity terms are turned off [18, 19]. Our derivation is quite general and valid not only for Fermi liquid as in Ref. [18, 19] but for any relativistic fermionic system. The phase space continuity can be shown to be broken by an anomalous term proportional to the product of vorticity and electric field. So the phase space measure is not conserved. It is modified by a factor related to the Berry curvature. We will also show that the conservation laws of the right- and left-hand currents is broken by anomalous terms, which can be given by the flux of a 4-dimensional monopole in Euclidean momentum space. Therefore a variety of properties such as CME/CVE, chiral anomaly, Berry curvature and 4-d Euclidean monopole can be described in a unified way in the quantum kinetic theory with Wigner functions.

Equation of motion with Berry curvature in 3-dimensions. — We will follow an example of Ref. [18] to illustrate the concept of Berry curvature. We consider a Hamiltonian $H' = \boldsymbol{\sigma}\cdot\mathbf{p}$ for spin-1/2 fermions in addition to the normal part $H(\mathbf{p},\mathbf{x})$, where $\boldsymbol{\sigma}$ are Pauli matrices. Under an adiabatic evolution, the action of the helicity plus state in path integral is

$$S = \int dt(\dot{\mathbf{x}}\cdot\mathbf{p} + \dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) - \dot{\mathbf{p}}\cdot\mathbf{a}(\mathbf{p}) - H(\mathbf{p},\mathbf{x})], \quad (1)$$

where $\mathbf{A}(\mathbf{x})$ is the electromagnetic vector potential and $\mathbf{a}(\mathbf{p})$ is the vector potential in momentum space resulting from diagonalizing H' in path integral. We can generalize the coordinate variables by combining $\mathbf{p}$ and $\mathbf{x}$, $\xi_a = (\mathbf{p}, \mathbf{x})$ with $a = 1, 2, \cdots, 6$. The action can be cast into a compact form,

$$S = \int dt[-\gamma_a(\xi)\dot{\xi}_a - H(\xi)], \tag{2}$$

where $\gamma_a = [\mathbf{a}(\mathbf{p}), -\mathbf{p} - \mathbf{A}(\mathbf{x})]$. The equation of motions follows the Lagrangian equations,

$$\gamma_{ab}\dot{\xi}_b = -\frac{\partial H(\xi)}{\partial \xi_a} \tag{3}$$

where $\gamma_{ab} \equiv \partial_a \gamma_b(\xi) - \partial_b \gamma_a(\xi)$ and given by

$$[\gamma_{ab}] = \begin{bmatrix} 0 & \Omega_3 & -\Omega_2 & -1 & 0 & 0 \\ -\Omega_3 & 0 & \Omega_1 & 0 & -1 & 0 \\ \Omega_2 & -\Omega_1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -B_3 & B_2 \\ 0 & 1 & 0 & B_3 & 0 & -B_1 \\ 0 & 0 & 1 & -B_2 & B_1 & 0 \end{bmatrix}, \tag{4}$$

where $\mathbf{\Omega} = \nabla_{\mathbf{p}} \times \mathbf{a}(\mathbf{p})$ is the Berry curvature and $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{x})$ is the 3-dimensional magnetic field. The determinant of $[\gamma_{ab}]$ is $\det[\gamma_{ab}] = (1 + \mathbf{\Omega} \cdot \mathbf{B})^2$. We see that the invariant phase space volume becomes $\sqrt{\det[\gamma_{ab}]}d^3\mathbf{x}d^3\mathbf{p}$, where $\sqrt{\det[\gamma_{ab}]} = |1 + \mathbf{\Omega} \cdot \mathbf{B}|$ indicates the change of phase space volume with time [23].

Quantum kinetic equation for Wigner functions. — In a quantum kinetic theory, the classical phase-space distribution $f(x, p)$ is replaced by the Wigner function $W(x, p)$ in space-time x and 4-momentum p, defined as the ensemble average of the Wigner operator [25–27] for spin-1/2 fermions,

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \bar{\psi}_\beta(x_+) U(x_+, x_-) \psi_\alpha(x_-), \tag{5}$$

where ψ_α and $\bar{\psi}_\beta$ are Dirac spinor fields, $x_\pm \equiv x \pm \frac{1}{2}y$ are two space-time points centered at x with space-time separation y, and the gauge link $U(x_+, x_-) \equiv \exp[-iQ\int_{x_-}^{x_+} dz^\mu A_\mu(z)]$ ensures the gauge invariance of $\hat{W}_{\alpha\beta}$. Here Q is the electromagnetic charge of the fermions, and A_μ is the electromagnetic vector potential. Note that we use the metric convention $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. To simplify the quantum kinetic equation under a background field we consider a massless and collisionless fermionic system in a constant external electromagnetic field $F_{\mu\nu}$ in the lab frame. Since we only consider an Abelian classical background field, we have dropped the path ordering in the gauge link. The Wigner function is a matrix in Dirac space and satisfies the quantum kinetic equation [25–27], $\gamma_\mu(p^\mu + \frac{i}{2}\nabla^\mu)W(x, p) = 0$, where γ^μ's are Dirac matrices and $\nabla^\mu \equiv \partial_x^\mu - QF^\mu{}_\nu \partial_p^\nu$. The Wigner function should contain information about quantum interactions and we will show that all currents including chiral anomaly can be derived from the above equation. The Wigner function can be decomposed in terms of 16 independent generators of the Clifford algebra whose coefficients are scalar, pseudo-scalar, vector, axial vector and tensor respectively. The quantum kinetic equation for the Wigner function then leads to two decoupled sets of equations for its components [25–27], one of which relevant to our study reads,

$$\begin{aligned} p^\mu \mathscr{V}_\mu &= 0, \quad p^\mu \mathscr{A}_\mu = 0, && (6) \\ \nabla^\mu \mathscr{V}_\mu &= 0, \quad \nabla^\mu \mathscr{A}_\mu = 0, && (7) \\ \epsilon_{\mu\nu\rho\sigma}\nabla^\rho \mathscr{A}^\sigma &= -2\left(p_\mu \mathscr{V}_\nu - p_\nu \mathscr{V}_\mu\right), && (8) \\ \epsilon_{\mu\nu\rho\sigma}\nabla^\rho \mathscr{V}^\sigma &= -2\left(p_\mu \mathscr{A}_\nu - p_\nu \mathscr{A}_\mu\right), && (9) \end{aligned}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita anti-symmetric tensor, with $\epsilon^{0123} = -\epsilon_{0123} = 1$. $\mathscr{V}_\mu(x, p)$ and $\mathscr{A}_\mu(x, p)$ are the vector and axial-vector component of the Wigner function, which will give rise to the vector and axial-vector current, respectively, after integrating over four-momentum.

We assume a system close to local equilibrium under a constant external field $F^{\mu\nu}$. Therefore, $\mathscr{V}_\mu(x, p)$ and $\mathscr{A}_\mu(x, p)$ will depend on x only through fluid four-velocity $u(x)$, temperature $T(x)$, chemical potential $\mu(x)$ and chiral chemical potential $\mu_5(x)$. We can determine the analytic form of the Wigner function in terms of $\{p, F^{\mu\nu}, u, T, \mu, \mu_5\}$ from Eqs. (6-9). We further assume that the space-time derivative ∂_x and the field strength $F_{\mu\nu}$ are small variables of the same order and can be used as parameters in the power expansion of $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$, so that one can use an iterative scheme to solve $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ order by order. The unique forms of $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ to the first order that satisfy Eqs. (6-9) was determined in Ref. [24] and reads,

$$\begin{aligned} \mathscr{X}^\mu &= p^\mu \delta(p^2) Z_0 + \frac{1}{2} p_\nu [u^\mu \omega^\nu - u^\nu \omega^\mu] \frac{\partial \bar{Z}_0}{\partial(u \cdot p)} \delta(p^2) \\ &\quad -Q p_\nu [u^\mu B^\nu - u^\nu B^\mu] \bar{Z}_0 \delta'(p^2) \\ &\quad +Q \epsilon^{\mu\lambda\rho\sigma} u_\lambda p_\rho E_\sigma \bar{Z}_0 \delta'(p^2), \end{aligned} \tag{10}$$

where $\mathscr{X} = (\mathscr{V}, \mathscr{A})$, $Z_0 = (V_0, A_0)$, $\bar{Z}_0 = (A_0, V_0)$, with the first order solutions V_0 and A_0 given by

$$\begin{aligned} [V_0, A_0] &= \sum_{s=\pm 1} \theta(su \cdot p) \left[(f_{s,R} + f_{s,L}), (f_{s,R} - f_{s,L})\right], \\ f_{s,\chi} &= \frac{2}{(2\pi)^3} \frac{1}{e^{s(u\cdot p - \mu_\chi)/T} + 1}, (\chi = R, L), \end{aligned} \tag{11}$$

where $R(L)$ denotes the right (left)-handed fermions and $\mu_{R,L} = \mu \pm \mu_5$ [2]. We have used notations $E_\sigma = u^\rho F_{\sigma\rho}$, $B_\sigma = (1/2)\epsilon_{\sigma\mu\nu\rho} u^\mu F^{\nu\rho}$ and $\omega_\mu = (1/2)\epsilon_{\mu\nu\rho\sigma} u^\nu \partial^\rho u^\sigma$, which depend on x via the fluid velocity $u(x)$. We use $\mathscr{X}_0^\mu$ to denote the zero-th order term $p^\mu \delta(p^2) Z_0$ in Eq. (10) and $\mathscr{X}_1^\mu$ to denote the rest, i.e. the first order terms. Note that solutions of $\mathscr{V}^\mu$ and $\mathscr{A}^\mu$ in Eq. (10) are static

ones where μ_5 and T are constants, $\mu = \text{const.} - QE\cdot x$ and $u^\sigma(\mathbf{x}-\mathbf{u}t/u^0)$ is in a solenoidal form with $\partial\cdot\mathbf{u}=0$.

We can derive the vector and axial-vector current and the energy-momentum tensor from $\mathscr{V}^\mu$ and $\mathscr{A}^\mu$ in Eq. (10) by integrating over momentum: $j^\mu = \int d^4p\mathscr{V}^\mu$, $j_5^\mu = \int d^4p\mathscr{A}^\mu$, and $T^{\mu\nu} = \frac{1}{2}\int d^4p(p^\mu\mathscr{V}^\nu + p^\nu\mathscr{V}^\mu)$. The current j^μ contains two parts proportional to magnetic field and vorticity, known as the CME and CVE [1–3, 9], respectively. So we have shown that both effects are direct consequences of the quantum kinetic equation for the Wigner function [24]. The axial-vector current j_5^μ also has two parts proportional to magnetic field and vorticity which can be regarded as a sort of dual effects to CME and CVE, respectively [3, 24]. Conservation equations for j^μ, j_5^μ (with anomaly) and $T^{\mu\nu}$ can then be derived by taking divergences of j^μ, j_5^μ and $T^{\mu\nu}$ [24]: $\partial_\mu j^\mu = 0$, $\partial_\mu j_5^\mu = -\frac{Q^2}{2\pi^2}E\cdot B$, and $\partial_\mu T^{\mu\nu} = QF^{\nu\rho}j_\rho$.

Relativistic chiral kinetic equation. — Now we try to derive a new form of relativistic chiral kinetic equation from Eq. (7). We start with Eq. (7) for the zero-th order Wigner function $\mathscr{Z}_0^\mu$,

$$\begin{aligned}\nabla_\mu\mathscr{Z}_0^\mu &= (\partial_\mu^x - QF_{\mu\nu}\partial_p^\nu)[p^\mu\delta(p^2)Z_0]\\ &= \delta(p^2)[p^\mu\partial_\mu^x - Qp^\mu F_{\mu\nu}\partial_p^\nu]Z_0 = 0,\end{aligned} \quad (12)$$

where Z_0 is the phase space distribution function and given in Eq. (11). Eq. (12) is a Vlasov-like equation, in which we can extract $dx^\sigma/d\tau = p^\sigma/m_0$ and $dp^\mu/d\tau = Qp^\mu F_{\mu\nu}/m_0$, where τ is the invariant time, m_0 is a mass scale (note: it is not the fermion mass since we are considering massless fermions), and $Qp^\mu F_{\mu\nu}/m_0$ is a general Lorentz force. It is interesting to see that the integration over p_0 in the local rest frame gives the Boltzmann equation without collision terms,

$$\begin{aligned}\int dp_0\nabla_\mu\mathscr{Z}_0^\mu &= \frac{1}{2}\sum_{s=\pm1}s[\partial_t + s\mathbf{v}\cdot\nabla_{\mathbf{x}}\\ &\quad +sq(\mathbf{E}+s\mathbf{v}\times\mathbf{B})\cdot\nabla_{s\mathbf{p}}]\\ &\quad \times(f_{s,R}\pm f_{s,L}).\end{aligned} \quad (13)$$

We see that the classical kinetic equation for fermions and anti-fermions can be derived and that the anti-fermion part has the same structure as the fermion one except the sign of charges and momenta.

Now we consider Eq. (7) for the first order Wigner function $\mathscr{Z}_1^\mu$,

$$\begin{aligned}\nabla_\mu\mathscr{Z}_1^\mu &= Q\delta(p^2)[(u\cdot b)B^\mu - (b\cdot B)u^\mu]\partial_\mu\bar{Z}_0\\ &\quad +Q\delta(p^2)\epsilon^{\mu\nu\rho\sigma}u_\nu b_\rho E_\sigma\partial_\mu\bar{Z}_0\\ &\quad -Q^2\delta(p^2)(E\cdot B)b^\sigma\partial_\sigma^p\bar{Z}_0 + Q\delta(p^2)\omega_\rho F^{\rho\sigma}\partial_\sigma^p\bar{Z}_0\\ &\quad +Q\delta(p^2)(p^\sigma F^{\xi\eta} - p^\xi F^{\sigma\eta})b_\eta\omega_\xi\partial_\sigma^p\bar{Z}_0,\\ &= 0\end{aligned} \quad (14)$$

where $b^\sigma \equiv -p^\sigma/p^2$. We will show that $\delta(p^2)b^\sigma$ is a 4-dimensional monopole in Euclidean momentum space. Combining Eq. (12) and Eq. (14), we obtain the relativistic chiral kinetic equation

$$\begin{aligned}&\frac{1}{2}\nabla_\mu(\mathscr{V}^\mu \pm \mathscr{A}^\mu) = 0\\ &\to \delta(p^2)\left[\frac{dx^\sigma}{d\tau}\partial_\sigma^x + \frac{dp^\sigma}{d\tau}\partial_\sigma^p\right]f_{R/L} = 0,\end{aligned} \quad (15)$$

where the upper/lower sign corresponds to the right/left-hand distribution, and $dx^\sigma/d\tau$ and $dp^\sigma/d\tau$ are given by

$$\begin{aligned}m_0\frac{dx^\sigma}{d\tau} &= p^\sigma \pm Q[(u\cdot b)B^\mu - (b\cdot B)u^\mu\\ &\quad +\epsilon^{\mu\nu\rho\sigma}u_\nu b_\rho E_\sigma],\\ m_0\frac{dp^\sigma}{d\tau} &= -Qp_\rho F^{\rho\sigma} \pm Q\omega_\rho F^{\rho\sigma} \pm Q(p^\sigma F^{\xi\eta} - p^\xi F^{\sigma\eta})\\ &\quad \times b_\eta\omega_\xi \mp Q^2(E\cdot B)b^\sigma.\end{aligned} \quad (16)$$

Here we have used notations

$$f_{R/L} \equiv \frac{1}{2}(V_0 \pm A_0) = \sum_{s=\pm1}\theta(su\cdot p)f_{s,R/L}. \quad (17)$$

In the local rest frame with $u^\sigma = (1,\mathbf{0})$, $B^\sigma = (0,\mathbf{B})$ and $E^\sigma = (0,\mathbf{E})$, the Berry curvature appears in Eq. (16) as $dx^0/d\tau = [p^0 \pm Q(\mathbf{b}\cdot\mathbf{B})]/m_0$ with $\mathbf{b}$ the spatial component of b^σ, which is similar to $1+\mathbf{\Omega}\cdot\mathbf{B}$ in Ref. [19]. Turning off the ω terms in Eq. (16), we obtain

$$\begin{aligned}\frac{d\mathbf{x}}{d\tau} &= \frac{1}{m_0}[\mathbf{p} \pm Q(b_0\mathbf{B} + \mathbf{E}\times\mathbf{b})]\\ \frac{d\mathbf{p}}{d\tau} &= \frac{Q}{m_0}[p_0\mathbf{E} + (\mathbf{p}\times\mathbf{B})] \pm \frac{Q^2}{m_0}(\mathbf{E}\cdot\mathbf{B})\mathbf{b},\end{aligned} \quad (18)$$

which is similar to Eqs. (14-15) in Ref. [19]. Part (but not all) of the ω terms can be generated by the replacement $QB^\sigma \to -(u\cdot p)\omega^\sigma$. The same relation is also observed in Ref. [19]. It is interesting to observe that there is a term $pmQ\omega_\rho F^{\rho\sigma}$ in $\frac{dp^\sigma}{d\tau}$ in Eq. (16). This shows that a vorticity behaves like a charged particle interacting only with the magnetic field but not the electric field.

Phase space continuity equation. — We now take spatial and momentum divergence of $dx^\sigma/d\tau$ and $dp^\sigma/d\tau$ respectively as

$$\begin{aligned}m_0\partial_\sigma\frac{dx^\sigma}{d\tau} &= 0,\\ m_0\partial_\sigma^p\frac{dp^\sigma}{d\tau} &= \mp\frac{2Q}{p^2}(u\cdot p)(\omega\cdot E) \pm \frac{2Q^2}{p^2}(E\cdot B),\end{aligned} \quad (19)$$

where we have used the conditions $u^\mu\partial_\mu\omega^\nu = u^\mu\partial_\mu B^\nu = 0$, $\partial_\mu\omega^\mu = 0$, $\partial_\mu B^\mu = 2(\omega\cdot E)$, $\partial^\mu u^\nu = \epsilon^{\mu\nu\tau\lambda}u_\tau\omega_\lambda$, $\epsilon^{\sigma\rho\eta\xi}u_\rho\partial_\sigma E_\xi = 0$, $\epsilon_{\mu\nu\sigma\rho}u^\mu\omega^\nu B^\sigma = 0$, $\epsilon^{\sigma\rho\alpha\beta}\epsilon_{\sigma\rho\mu\nu} = -2\delta^{\alpha\beta}_{[\mu\nu]}$, and $\partial_\sigma^p b^\sigma = -2/p^2$. It is interesting to see that $dx^\sigma/d\tau$ is conserved but $dp^\sigma/d\tau$ is not. Using Eq. (15) and $\partial_\sigma^p[\delta(p^2)] = 2b_\sigma\delta(p^2)$, we arrive at the phase space

continuity equation

$$\partial_\sigma\left[\frac{dx^\sigma}{d\tau}\delta(p^2)f_{R/L}\right]+\partial^p_\sigma\left[\frac{dp^\sigma}{d\tau}\delta(p^2)f_{R/L}\right]$$
$$=\pm\frac{2Q}{m_0}(u\cdot b)(\omega\cdot E)\delta(p^2)f_{R/L}. \quad (20)$$

Note that there is an anomalous source term proportional to $\omega\cdot E$.

Anomaly and 4-dimensional monopole. — From Eq. (10), the anomalous conservation law of the left- and right-hand current can be derived as

$$\begin{aligned}\partial_\mu j^\mu_{R/L} &= \mp 2\pi Q^2(E\cdot B)\int d(u\cdot p)\delta(u\cdot p)f_{R/L}\\ &= \mp\frac{Q^2}{4\pi^2}(E\cdot B)\end{aligned} \quad (21)$$

where $j^\mu_{R/L}=(j^\mu\pm j^\mu_5)/2$, and we have used from Eq. (17): $f_{R/L}(u\cdot p=0)=1/(2\pi)^3$. In deriving Eq. (21), we completed all integrals in Minkowski space. In order to see the connection to the 4-dimensional monopole and the relativistic chiral kinetic equation, we can carry out the integral in Euclidean space as follows

$$\begin{aligned}\partial_\mu j^\mu_{R/L} &= \pm Q^2(E\cdot B)\int d^4p\delta(p^2)b^\sigma\partial^p_\sigma f_{R/L}\\ &= \pm Q^2(E\cdot B)\frac{1}{\pi}\mathrm{Im}\int d^4p\frac{1}{p^2+i\epsilon}\frac{1}{p^2}p^\sigma\partial^p_\sigma f_{R/L}\\ &= \pm Q^2(E\cdot B)\frac{1}{\pi}\mathrm{Im}\left[i\int d^4p_E\right.\\ &\quad\left.\times\frac{1}{p_E^2-i\epsilon}\frac{1}{p_E^2}p_E^\sigma\partial^{p_E}_\sigma f_{R/L}\right]\\ &= \mp Q^2(E\cdot B)\frac{1}{\pi}\mathrm{Im}\left[i\int d^4p_E\right.\\ &\quad\left.\times\partial^{p_E}_\sigma\left(\frac{p_E^\sigma}{p_E^4-i\epsilon}\right)f_{R/L}\right]\\ &= \mp\frac{Q^2}{4\pi^2}(E\cdot B).\end{aligned} \quad (22)$$

Note that the anomalous term comes from the $E\cdot B$ term in $dp^\sigma/d\tau$ in Eq. (16). We have taken analytic continuation $p_0=ip_4$ and $p^2=-p_E^2$. Note that in the second equality of Eq. (22) the poles in Minkowski space are $p_0=\pm\sqrt{|\mathbf{p}|^2-i\epsilon}=\pm|\mathbf{p}|\mp i\epsilon$, in order to avoid these poles in Wick rotation the integral limit of p_0 should be $[-i\infty,i\infty]$ which corresponds to $[-\infty,\infty]$ for the p_4 integral. We have also used

$$\partial^{p_E}_\sigma(p_E^\sigma/p_E^4)=2\pi^2\delta^{(4)}(p_E^\sigma). \quad (23)$$

Although for $p_E\neq 0$, we have $\partial^{p_E}_\sigma(p_E^\sigma/p_E^4)=0$, but the integral is non-vanishing, since

$$\int d^4p_E\partial^{p_E}_\sigma(p_E^\sigma/p_E^4) = \oint dS_{3,\sigma}p_E^\sigma/p_E^4=2\pi^2. \quad (24)$$

Note that the n-volume of n-sphere or the hyper-surface area of (n+1)-ball with radius R is given by $S_n=[2\pi^{(n+1)/2}/\Gamma((n+1)/2)]R^n$. So we see that $\delta(p^2)b^\sigma$ plays the role of a 4-dimensional monopole in Euclidean momentum space, where the singular point of the monopole is located at $p=0$. This is similar to the 3-dimensional case where the singular point is at $\mathbf{p}$ [18, 19].

In summary, we have shown that the Berry curvature and a 4-dimensional monopole in Euclidean momentum space emerge in a new chiral kinetic equation with manifest Lorentz covariance. The chiral anomaly has been shown to be given by the flux of this 4-dimensional monopole. The phase space continuity equation has also been derived but has an anomalous source term proportional to the contraction of vorticity and electric field. The derivation of the chiral kinetic equation is quite general and valid for relativistic fermionic system. It would be interesting to investigate the effect of the collision terms in the future.

Note added: During the completion of this work, we learned that Son and Yamamoto were also working on the similar topic [28].

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[1] D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A **803**, 227 (2008).
[2] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[3] D. E. Kharzeev and D. T. Son, Phys. Rev. Lett. **106**, 062301 (2011).
[4] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, JHEP **0901**, 055 (2009).
[5] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, JHEP **1101**, 094 (2011).
[6] M. Torabian and H. U. Yee, JHEP **0908**, 020 (2009).
[7] A. Rebhan, A. Schmitt and S. A. Stricker, JHEP **1001**, 026 (2010).
[8] T. Kalaydzhyan and I. Kirsch, Phys. Rev. Lett. **106**, 211601 (2011).
[9] D. T. Son and P. Surowka, Phys. Rev. Lett. **103**, 191601 (2009).
[10] S. Pu, J. H. Gao and Q. Wang, Phys. Rev. D **83**, 094017 (2011).
[11] D. E. Kharzeev and H. -U. Yee, Phys. Rev. D **84**, 045025 (2011).
[12] M. A. Metlitski and A. R. Zhitnitsky, Phys. Rev. D **72**, 045011 (2005).
[13] G. M. Newman and D. T. Son, Phys. Rev. D **73**, 045006 (2006).
[14] J. Charbonneau and A. Zhitnitsky, JCAP **1008**, 010

(2010).
[15] M. Lublinsky and I. Zahed, Phys. Lett. B **684**, 119 (2010).
[16] M. Asakawa, A. Majumder and B. Muller, Phys. Rev. C **81**, 064912 (2010).
[17] K. Landsteiner, E. Megias and F. Pena-Benitez, Phys. Rev. Lett. **107**, 021601 (2011).
[18] D. T. Son and N. Yamamoto, arXiv:1203.2697 [cond-mat.mes-hall].
[19] M. A. Stephanov and Y. Yin, arXiv:1207.0747 [hep-th].
[20] C. H. Wong and Y. Tserkovnyak, Phys. Rev. B **84**, 115209 (2011).
[21] M. V. Berry, Proc. Roy. Soc. Lond. **A392**, 45 (1984).
[22] D. Xiao, M.-C. Chang, and Q. Niu, Rev. Mod. Phys. **82**, 1959 (2010).
[23] D. Xiao, J. -r. Shi and Q. Niu, Phys. Rev. Lett. **95**, 137204 (2005) [cond-mat/0502340].
[24] J. -H. Gao, Z. -T. Liang, S. Pu, Q. Wang and X. -N. Wang, arXiv:1203.0725 [hep-ph].
[25] D. Vasak, M. Gyulassy and H. T. Elze, Annals Phys. **173**, 462 (1987).
[26] H. T. Elze, M. Gyulassy and D. Vasak, Nucl. Phys. B **276**, 706 (1986).
[27] H. T. Elze and U. W. Heinz, Phys. Rept. **183**, 81 (1989).
[28] D. T. Son and N. Yamamoto, private communication.